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Some Properties of Analytic in a Disk Functions with Applications to the Study of the Behavior of Singular Integrals

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Abstract—We obtain representations for an analytic in a disc function such that its real part has a zero of an integer order at a fixed boundary point. We consider certain applications of these representations for studying properties of singular integrals with Hilbert kernel.

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1. We study functions analytic in the disk $|z| < 1$ in the plane of the complex variable $z = x + iy$. Denote a boundary point of the disk by $t = e^{i\gamma}$, $0 \le \gamma \le 2\pi$. We consider the Schwarz boundary-value problem that implies the evaluation of an analytic in the mentioned disk function $w(z)$ satisfying the boundary condition

$$\operatorname{Re} w(t) = d(t)|t-1|^{2n}, \tag{1}$$

where n is any given positive integer, $d(t)$ is a given function satisfying the Hölder condition, i.e., Condition H ([1], P. 8), on the circle $|t| = 1$, $d(1) \neq 0$. We study the behavior of the function $w(z)$ near the point $z = 1$. In particular, to this end we can use the Schwarz formula which gives a solution to problem (1) (e.g., [2], P. 58), putting $\operatorname{Im} w(1) = 0$. But this way is rather difficult. The immediate application of the known Plemelj–Privalov results on the behavior of boundary values of the Cauchy-type integral and the behavior of singular integrals with Cauchy or Hilbert kernels near a fixed point of the integration path ([1], pp. 58, 61, 160) to studying the Schwarz integral with boundary condition (1) gives no informative conclusions.

Therefore we apply another approach which is based on the representation of condition (1) in an equivalent form.

For $t = e^{i\gamma}$, $0 \le \gamma \le 2\pi$, we have

$$t - 1 = \left(2\sin\frac{\gamma}{2}\right)e^{i(\pi+\gamma)/2}, \tag{2}$$

hence

$$\frac{-t}{(t-1)^2} = \frac{1}{\left(2\sin\frac{\gamma}{2}\right)^2} = \frac{1}{|t-1|^2}. \tag{3}$$

In these terms we rewrite the boundary value condition (1) in the form

$$\operatorname{Re}\left[w(t)\left(\frac{-t}{(t-1)^2}\right)^n\right] = d(t), \quad t \neq 1. \tag{4}$$

By the Schwarz formula we determine the function

$$J(z) = \frac{1}{2\pi}\int_0^{2\pi} d(e^{i\sigma})\frac{e^{i\sigma}+z}{e^{i\sigma}-z}d\sigma, \tag{5}$$