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Bisectorial Operator Pencils and the Problem of Bounded Solutions

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Abstract—We consider a linear differential equation unresolved with respect to the derivative. We assume that the spectrum of the corresponding pencil is contained in two sectors. We study the unique existence of a bounded solution with any bounded free term.

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It is well-known [1–3] that the equation $u' - Au = f$, where A is a linear bounded operator, has a unique solution bounded on the axis with any bounded right-hand side f if and only if the spectrum of A does not intersect the imaginary axis.

One can easily extend a particular case of this assertion, when the whole spectrum lies in the left half-plane, to the case when the coefficient A is unbounded and satisfies the sectorial condition [4–8]. This condition means that the spectrum lies in some sector in the left half-plane, and the resolvent satisfies some growth estimate at infinity.

In this paper we consider the case of a bisectorial pencil, i.e., the case, when the spectrum belongs to two sectors which lie in the left and right half-planes, correspondingly (see the left Fig. 1a)). The notion of a bisectorial pencil (in the original terminology, a bisemigroup) $\lambda \mapsto \lambda\mathbf{1} - A$ (here the symbol $\mathbf{1}$ denotes the identity operator) generated by an unbounded operator A was introduced in [9] and studied in [10]. Here we consider a more general bisectorial pencil $\lambda \mapsto \lambda F - G$. We assume that operators F and G act from a Banach space X to a Banach space Y and are bounded; usually one can reduce the case of unbounded F and G to the case of bounded ones [11].

Let us mention two specific features of the problem under consideration. In the case of a sectorial pencil and the equation $u' - Au = f$, when the coefficient A acts from its domain of definition $D(A) \subset X$ to X, the corresponding semigroup of operators $T(t)$, $t \geq 0$, acts from X to X (rather than to $D(A)$). That is why values of the solution $u(t) = \int_0^{+\infty} T(s) f(t-s)\, ds$ belong to X (rather than to $D(A)$). In the case of a bisectorial pencil $\lambda \mapsto \lambda F - G$ and the equation $Fu' - Gu = f$, the space X, where F and G are defined, appears to be an analog of $D(A)$; therefore, if values of the function f belong to Y, then the range of the solution is wider than X. For the equation $Fu' - Gu = f$ the determination of a wider suitable space containing X is a separate problem. In this connection we consider here functions f whose values belong to some subspace Y^1 of the space Y. This indirectly means that X turns into a space that includes the domain of definition of F and G. Auxiliary considerations (see Example 2 in Section 4) allow us to apply the theorem proved in this paper even if values of f run over the whole space Y; in this case the set of values of u is wider than X.

The next specific feature is the following property. If for ordinary differential equations and those with an unbounded sectorial operator the Green function $t \mapsto \mathcal{G}(t)$ has a discontinuity of the first kind at zero, then for an arbitrary bisectorial pencil there occurs an integrable discontinuity of the second kind (Proposition 5).

The subject studied in [12] is most close to that of our paper. One proves in [12] that if there exists an integrable Green function, then with any bounded right-hand side there exists a unique bounded

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