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Distribution of Points of One-Dimensional Quasilattices with Respect to a Variable Module

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Abstract—We consider one-dimensional quasiperiodic Fibonacci tilings. Namely, we study sets of vertices of these tilings that represent one-dimensional quasilattices defined on the base of a parameterization by rotations of a circle, and the distribution of points of quasilattices with respect to a variable module. We show that the distribution with respect to some modules is not uniform. We describe the distribution function and its integral representation, and estimate the remainder in the problem of the distribution of points of a quasilattice for corresponding modules.

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1. ONE-DIMENSIONAL QUASILATTICES. THE UNIFORM DISTRIBUTION

At present times in the theory of numbers, together with the study of periodic structures (like progressions, lattices, or periodic tilings), one actively investigates non-periodic structures, in particular, quasiperiodic ones. A new direction in the theory of numbers, namely, the theory of one-dimensional quasiperiodic structures [1–9] is being intensively developed; this theory has many applications in the physics of quasicrystals [10, 11]. A classic example of an one-dimensional quasiperiodic structure is presented by infinite Fibonacci tilings [1, 2]. These tilings can be defined in different ways [3, 4]. For example, in the paper [5] (P. 143) they are defined by means of the intersection (in a plane) of the ray $y = \alpha x$, where the inclination angle α is irrational, and the integer lattice formed by vertical and horizontal lines intersecting at points with integer coordinates. As a result we obtain an infinite word consisting of zeros and units, namely, the Sturm sequence. It is constructed by the following rule: zeros correspond to cases when the line $y = \alpha x$ intersects a vertical line of the lattice, while units do to cases when the line $y = \alpha x$ intersects a horizontal line of the lattice. We associate a zero in this word with a half-interval of length l_1, and we do a unit with a half-interval of length l_2. Then as a result of the successive non-commutative application of these half-intervals we obtain an infinite Fibonacci tiling. The left endpoints of half-intervals of the tiling form a sequence, which is said to be an one-dimensional quasilattice $\{x_n\}$. It can be defined [6] by the rule: 1) $x_{-1} = 0$, 2) the transition from x_n to x_{n+1} is realized by the formula

$$x_{n+1} = \begin{cases} x_n + l_1, & \text{if } \langle n\alpha\rangle \in [0; 1-\alpha); \\ x_n + l_2, & \text{if } \langle n\alpha\rangle \in [1-\alpha; 1), \end{cases}$$

where the symbol $\langle\cdot\rangle$ is the fractional part, α is some irrationality, while l_1 and l_2 are arbitrary real numbers.

Let us adduce the main definitions which are necessary in what follows. Let $E = [a, b) \subseteq [0, 1)$. Denote the integer part modulo h as $[x]_h = h\left[\frac{x}{h}\right]$, and $\langle x\rangle_h = x - [x]_h$. Introduce the function

$$N_h(E, n, \{x_n\}) = \sharp\{0 \le i < n : \langle x_i\rangle_h \in E\}.$$

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