ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 10, pp. 82–85. © Allerton Press, Inc., 2011.
Original Russian Text © A.M. Bikchentaev, 2011, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2011, No. 10, pp. 94–98.

# The Haagerup Problem on Subadditive Weights on $W^*$-Algebras

**A. M. Bikchentaev**[1*]
(Submitted by D.Kh. Mushtari)

[1] *Kazan (Volga Region) Federal University, ul. Professora Nuzhina 1/37, Kazan, 420008 Russia*

Received March 23, 2011

**Abstract**—In 1975 U. Haagerup stated the following question: *Is every normal subadditive weight on a $W^*$-algebra sigma-weakly lower semicontinuous*? Here we positively answer this question in a particular case of abelian $W^*$-algebras and present a general form of normal subadditive weights on these algebras.

**DOI:** 10.3103/S1066369X11100112

Keywords and phrases: *$C^*$-algebra, $W^*$-algebra, subadditive weight, measurable function.*

## INTRODUCTION

As is noted in the abstract, in this paper we solve the U. Haagerup problem [1] for the case of abelian $W^*$-algebras. We also obtain an application to decompositions of sublinear expectations [2] on measurable functions. The main results (without proofs) are announced in [3].

## 1. SUBADDITIVE WEIGHTS ON A $C^*$-ALGEBRA

Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{A}^+$ be the cone of nonnegative elements of $\mathcal{A}$. In [1] (see also [4], remark 1.6) the following definition is given.

**Definition 1.1.** A *subadditive weight* on a $C^*$-algebra $\mathcal{A}$ is a mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ such that

(i) $x \le y \Longrightarrow \varphi(x) \le \varphi(y)$ for all $x, y \in \mathcal{A}^+$,

(ii) $\varphi(x + y) \le \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{A}^+$,

(iii) $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x \in \mathcal{A}^+$, $\lambda \ge 0$ (here $0 \cdot (+\infty) \equiv 0$).

**Theorem 1.1.** *Let $X$ and $Y$ be partially ordered Banach spaces with positive cones $X^+$ and $Y^+$ and monotone norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let a mapping $T : X^+ \to Y^+$ be such that*

1) $0 \le_X x \le_X y \Rightarrow T(x) \le_Y T(y)$ $(x, y \in X)$;

2) $T(\lambda x) = \lambda T(x)$ $(\lambda \ge 0,\ x \in X^+)$.

*Then there exists $C > 0$ such that $\|T(x)\|_Y \le C\|x\|_X$ for all $x \in X^+$.*

**Proof.** Assume the contrary. Then for each $n \in \mathbb{N}$ there exists an element $x_n \in X^+$ such that $\|x_n\|_X = 2^{-n}$ and $\|T(x_n)\|_Y \ge n$. Therefore

$$x = \sum_{n=1}^{\infty} x_n \ge_X x_n$$

for all $n \in \mathbb{N}$ and $\|T(x)\|_Y \ge \|T(x_n)\|_Y \ge n$, which is a contradiction.

*E-mail: Airat.Bikchentaev@ksu.ru.