ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 10, pp. 59–69. © Allerton Press, Inc., 2011.
Original Russian Text © M.V. Falaleev and S.S. Orlov, 2011, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2011, No. 10, pp. 68–79.

Degenerate Integro-Differential Operators in Banach Spaces and Their Applications

M. V. Falaleev* **and S. S. Orlov****

Irkutsk State University, ul. K. Marksa 1, Irkutsk, 664003 Russia

Received August 24, 2010

Abstract—In this paper we consider linear integro-differential equations in Banach spaces with Fredholm operators at the highest-order derivatives and convolution-type Volterra integral parts. We obtain sufficient conditions for the unique solvability (in the classical sense) of the Cauchy problem for the mentioned equations and illustrate the abstract results with pithy examples. The studies are carried out in classes of distributions in Banach spaces with the help of the theory of fundamental operator functions of degenerate integro-differential operators. We propose a universal technique for proving theorems on the form of fundamental operator functions.

DOI: 10.3103/S1066369X11100082

Keywords and phrases: *Banach spaces, Fredholm operator, Jordan sets, distributions, fundamental operator functions.*

INTRODUCTION

In many works [1–7] devoted to degenerate differential equations, one has demonstrated the effectiveness of the application of fundamental operator functions to studying initial boundary-value problems for wide classes of differential operators with various types of singularities (Fredholm, Noetherian, spectral, sectorial, and the radial boundedness of operator bundles). But fundamental operator functions have not been constructed yet for integro-differential operators, in this paper we make an attempt to fill up this gap. We are first to propose a universal technique for proving theorems about fundamental operator functions for integro-differential operators. Along with a purely theoretical interest, degenerate integro-differential equations with Fredholm operators in the principal part play an important role in solving some initial boundary-value problems in the mathematical theory of visco- and thermoelasticity [8–10].

1. AUXILIARY ASSERTIONS

1.1. *Generalized Jordan sets of Fredholm operators.* In what follows the symbol B stands for a Fredholm operator, i.e., a closed linear operator acting from a Banach space E_1 to a Banach space E_2 that has a dense definition domain $\overline{D(B)} = E_1$ and a closed range $\overline{R(B)} = R(B)$, $\dim N(B) = \dim N(B^*) = n$. Assume that $\{\varphi_i\}_{i=1}^n$ is a basis in $N(B)$; $\{\psi_i\}_{i=1}^n$ is a basis in $N(B^*)$; $\{\gamma_i\}_{i=1}^n \subset E_1^*$ and $\{z_i\}_{i=1}^n \subset E_2$ are biorthogonal systems corresponding to these bases, i.e.,

$$\langle \varphi_i,\, \gamma_j\rangle = \langle z_i, \psi_j\rangle = \delta_{ij},\ \ i,\ j = 1, \dots, n.$$

Then, as is shown in [11] (P. 340), the operator $\widetilde{B} = B + \sum\limits_{i=1}^{n} \langle \cdot, \gamma_i\rangle\, z_i$ is continuously invertible; we denote the inverse one (it is called the Trenogin–Schmidt operator) by $\Gamma \in \mathcal{L}(E_2, E_1)$. Here $\Gamma z_i = \varphi_i$ and $\Gamma^* \gamma_i = \psi_i$.

*E-mail: mihail@ic.isu.ru.
**E-mail: orlov_sergey@inbox.ru.