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Finite Nilpotent Groups with Relatively Large Centralizers of Noninvariant Subgroups

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Abstract—We study groups satisfying the following condition: for any noninvariant subgroup A, the index of the product of A and the centralizer of A in its normalizer divides a fixed (for a given group) prime number. We give the complete description ot two-step nilpotent p-groups with the above indicated property.

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In the paper, we study finite nilpotent groups such that, for any noninvariant subgroup A,

$$|N(A) : A \cdot C(A)| \text{ divides } p, \qquad (*)$$

where p is a prime number fixed for a given group, i.e., $|N(A) : A \cdot C(A)| \in \{1; p\}$.

In [1], it has been shown that if, in a nonabelian periodic nilpotent group G, for any noninvariant subgroup A, we have $N(A) = A \cdot C(A)$, then $|G'|$ is a prime number. In [2], the case $p = 2$ has been studied. By this reason, in what follows p is an odd prime number.

Theorem 1. *In a finite nilpotent group G, condition $(*)$ holds for any noninvariant subgroup A if and only if $G = P \times H$, where P is a Silov p-subgroup of G in which condition $(*)$ holds for any noninvariant subgroup and the Hall p'-subgroup H is either abelian or has commutant of prime order, in the second case P is an abelian group.*

Proof. Necessity. Let $G = P \times H$, where P is a Silov p-subgroup of G. Since, for any noninvariant subgroup A of the group H, the equality $N_H(A) = A \cdot C_H(A)$ holds, then, as was mentioned above, from the results of [1] it follows that H is either abelian or has commutant of prime order.

Assume that P is a nonabelian group. Then from the fact that p is an odd number it follows that P cannot be a Dedekind group, i.e., P has a noninvariant subgroup K. If B is a maximal abelian subgroup of the group H and $A = K \times B$, then the subgroup A is not invariant in G. At the same time, since H is either abelian group or has commutant of prime order, it follows that $B \triangleleft H$. But then $H \leqslant N(A)$. From $(*)$, we obtain $H \leqslant A \cdot C(A)$. Hence $H = B \cdot C_H(B) = B$, i.e., H is an abelian subgroup.

Sufficiency. Let A be an arbitrary noninvariant subgroup of G, $K = A \cap P$, and $B = A \cap H$. Since $A \ntriangleleft G$, either $K \ntriangleleft P$ or $B \ntriangleleft H$. If H is an abelian group, then $N(A) = N_P(K) \times H$ and $C(A) = C_P(K) \times H$, which implies that the index

$$|N(A) : A \cdot C(A)| = |N_P(K) : K \cdot C_P(K)|$$

divides p. If H is a nonabelian subgroup, then P is an abelian subgroup and, from the conditions $N(A) = P \times N_H(B)$ and $C(A) = P \times C_H(B)$, we obtain

$$|N(A) : A \cdot C(A)| = |N_H(B) : B \cdot C_H(B)|.$$

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