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A Continuous Regularization Method of the First Order for Nonlinear Monotone Equations

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Assume that X is a uniformly convex and uniformly smooth Banach space, X^* is its conjugate one, $A : X \to X^*$ is a continuous monotone operator, $D(A) = X$, the equation

$$Ax = f \tag{0.1}$$

has a nonempty solution set N in X, x^* is the normal solution to (1) (a solution with the minimal norm). Under these assumptions one cannot establish a continuous dependence of a solution to (1) on perturbances of A and f. Therefore one should consider the problem on finding a solution to equation (1) as an ill-posed one and solve it with the help of a certain regularization method. Recently continuous solution methods for ill-posed problems arouse much interest. These methods are reduced to the solution of the Cauchy problem for a differential equation of a certain order. The order of the differential equation is said to be the order of a continuous method. For the case, when $X = H$ is a Hilbert space, continuous methods are investigated rather well [1]–[6]. In [7], [8], one studies the convergence of a continuous method of the first order for equation (1) with a monotone and accretive operator A in a Banach space, assuming that the operator A is differentiable. Note that in the investigation of the convergence of a continuous method in a Banach space not only the properties of the operator A, but also the geometric properties of the spaces X and X^* play an essential role. The objective of this paper is to establish the conditions which are sufficient for the convergence of a continuous method of the first order in a Banach space without the assumption on the differentiability of the monotone operator A.

Let $\delta_X(s)$ be the module of convexity of the space X. Assume that this function is continuous and grows on $[0, 2]$, $\delta_X(0) = 0$ ([9], p. 49; [10]). Consequently, the inverse function $\delta_X^{-1}(\varepsilon)$ exists.

Since the space X is uniformly smooth, we have that X^* is also uniformly convex ([9], p. 34). Let $\delta_{X^*}(s)$ be the module of convexity of X^*. Define the function $g_{X^*}(s) = \delta_{X^*}(s)/s$. It is well known [10] that this function is continuous and grows on $[0, 2]$, $g_{X^*}(0) = 0$. Thus we can construct the function $g_{X^*}^{-1}(\varepsilon)$.

Assume that the operator of the dual mapping $J : X \to X^*$ is defined by the correlations ([11], p. 311)

$$\|Jx\| = \|x\|, \quad \langle Jx, x\rangle = \|x\|^2 \ \ \forall x \in X. \tag{0.2}$$

The properties of this operator are defined by the geometric properties of the spaces X and X^*. Note that under the above assumptions about the space X the operator J is monotone, bounded, and continuous ([11], pp. 313, 330, 331). In addition, the following inequality is true (see [8], [12])

$$\|Jx - Jy\| \le \overline{c}_2 g_{X^*}^{-1}(2\overline{c}_2 L\|x - y\|), \tag{0.3}$$

where $\overline{c}_2 = 2\max\{1, \|x\|, \|y\|\}$, L is the Figel constant, $1 < L < 3.18$.

Introduce the functional [12]

$$V(x, y) = \|Jx\|^2/2 - \langle Jx, y\rangle + \|y\|^2/2 \ \ \forall x, y \in X.$$

It is well known [12], [8], that the functional $V(x, y)$ is nonnegative, convex, and differentiable with respect to Jx and y; in addition,

$$\operatorname{grad}_{Jx} V(x, y) = x - y, \quad \operatorname{grad}_y V(x, y) = Jy - Jx. \tag{0.4}$$