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# The Study of the Main Boundary Value Problems for One Degenerate Elliptic Equation by the Method of Potentials

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In this paper, we construct a fundamental solution and simple and double layer potentials for a degenerate elliptic equation of the second order. Using these potentials, we reduce the boundary value problems to the integral Fredholm equations of the second kind.

## 1. THE GREEN FORMULAE

Let $\mathrm{E}_p^+$ be the half-space $x_p > 0$ of the $p$-dimensional Euclidian space of points $x = (x', x_p)$, $x' = (x_1, x_2, \ldots, x_{p-1})$; let $D$ be a finite domain in $\mathrm{E}_p^+$ confined by the open part $\Gamma_0$ of the hyperplane $x_p = 0$ and a hypersurface $\Gamma$.

Consider in $\mathrm{E}_p^+$ the degenerate elliptic equation

$$E[U(x)] = \sum_{j=1}^{p-1} \frac{\partial^2 U}{\partial x_j^2} + \frac{\partial}{\partial x_p}\left(x_p^\alpha \frac{\partial U}{\partial x_p}\right) = 0, \tag{1.1}$$

where $0 < \alpha < 1$, $p \geq 3$.

Using the method of separation of variables, one can prove that equation (1.1) has a solution, tending to zero as $x_p \to 0$.

Let the functions $U, V \in C^2(D) \cap C^1(\overline{D})$. Direct computations enable one to verify the identity

$$VE[U] + \left(\sum_{j=1}^{p-1} \frac{\partial U}{\partial x_j}\frac{\partial V}{\partial x_j} + x_p^\alpha \frac{\partial V}{\partial x_p}\frac{\partial U}{\partial x_p}\right) = \sum_{j=1}^{p-1} \frac{\partial}{\partial x_j}\left(V \frac{\partial U}{\partial x_j}\right) + \frac{\partial}{\partial x_p}\left(x_p^\alpha V \frac{\partial U}{\partial x_p}\right). \tag{1.2}$$

Integrating both parts of identity (1.2) over the domain $D$ and using the Ostrogradskii formula, we obtain

$$\int_D VE[U]dx + \int_D \left(\sum_{j=1}^{p-1} \frac{\partial U}{\partial x_j}\frac{\partial V}{\partial x_j} + x_p^\alpha \frac{\partial V}{\partial x_p}\frac{\partial U}{\partial x_p}\right)dx = \int_\Gamma VA[U]d\Gamma. \tag{1.3}$$

Here

$$A[U] = \sum_{j=1}^{p-1} \cos(n, x_j)\frac{\partial U}{\partial x_j} + x_p^\alpha \cos(n, x_p)\frac{\partial U}{\partial x_p}$$

is the conormal derivative, $n$ is an outer normal to $\Gamma$. Formula (1.3) is called the first Green formula for the operator $E$. Interchanging $U$ and $V$ in formula (1.3), we get

$$\int_D UE[V]dx + \int_D \left(\sum_{j=1}^{p-1} \frac{\partial V}{\partial x_j}\frac{\partial U}{\partial x_j} + x_p^\alpha \frac{\partial U}{\partial x_p}\frac{\partial V}{\partial x_p}\right)dx = \int_\Gamma UA[V]d\Gamma.$$