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Approximation of Solutions to Nonregular Nonlinear Equations by Attractors of Dynamic Systems in a Banach Space

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1. PROBLEM DEFINITION

Consider the equation

$$F(x) = 0, \quad x \in X, \tag{1.1}$$

where $F : X \to X$ is a nonlinear operator, acting in a Banach space X. Assume that the space X is reflexive and has a Gateaux differentiable norm. The simplest examples of such spaces are l_p, L_p with $1 < p < \infty$, as well as the Hilbert spaces ([1], pp. 35–37). Denote by x^* a solution to equation (1.1) and put $\Omega_R(x) = \{z \in X : \|z - x\| < R\}$. Assume that the operator F is Frechet differentiable and

$$\|F'(x_1) - F'(x_2)\| \le L\|x_1 - x_2\| \ \ \forall x_1, x_2 \in \Omega_R(x^*). \tag{1.2}$$

Hereinafter the symbol $\|\cdot\|$ stands for the norms of various spaces. The derivative $F'(x)$ is not necessarily boundedly invertible, so the initial problem (1.1) is, generally, ill-posed. Heeding the fact that $F(x)$ is defined inaccurately, a stable approximation of solutions to the problem is based on regularization methods (see, e. g., [2], pp. 126–128; [3], pp. 9–11; [4], pp. 15–18). Assume that instead of the exact operator F in (1.1) we know only its approximation $\widetilde{F} : X \to X$; let the latter be Frechet differentiable and satisfy inequality (1.2) (with the same constant L) and the conditions

$$\|\widetilde{F}(x^*)\| \le \delta; \quad \|\widetilde{F}'(x) - F'(x)\| \le \delta \ \ \forall x \in \Omega_R(x^*). \tag{1.3}$$

A fruitful and widely used in the computational practice approach [5] to the development of methods, approximating the solution x^*, implies the construction of a discrete or continuous dynamic system such that x^* is its point of attraction. In a continuous case, in order to realize this idea, one has to construct on a phase space X a dynamic system

$$\dot{x} = \Phi(x), \quad x = x(t), \quad t \ge 0 \quad (\Phi : X \to X), \tag{1.4}$$

such that x^* is its asymptotically stable stationary point, i. e.,

$$\Phi(x^*) = 0, \quad \lim_{t\to+\infty} \|x(t) - x^*\| = 0$$

at least for the initial values $x(0) = x_0$ from a fixed neighborhood of x^*. In the regular case, when the operator $F'(x)$ or $F'^*(x)F'(x)$ is continuously invertible, one can take as (1.4), for example, the systems

$$\begin{gathered} \dot{x} = -F'^*(x)F(x), \quad \dot{x} = -F'(x)^{-1}F(x), \\ \dot{x} = -(F'^*(x)F'(x))^{-1}F'^*(x)F(x). \end{gathered} \tag{1.5}$$

The latter represent continuous analogs of the classical iterative gradient descent method, the Newton–Kantorovich methods, and Gauss–Newton ones. The second system is defined in an arbitrary Banach space, the rest two ones are used in Hilbert spaces and admit a natural generalization for equations