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Invertible Linear Relations Generated by a Uniformly Well-Posed Problem and a Nonnegative Operator Function

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1. In this paper, we describe invertible restrictions of the maximal relation generated by a nonnegative operator function and a first order differential expression for which the Cauchy problem is uniformly well-posed.

Note that the general definition of a relation generated by a pair of linear operators was given in [1]. In a series of papers (see, e. g., [2]–[6]), various problems of the spectral theory of linear relations generated by a nonnegative matrix function and various formally self-adjoint differential expressions were studied. In particular, in [3]–[6], resolutions and generalized resolutions were described. In [2]–[4] and partially in [5], the basic space was supposed to be finite-dimensional. In the infinite-dimensional case, the situation becomes complicated not only by the fact that some of the operator coefficients may be unbounded operators. One of the difficulties which arises in the infinite-dimensional case lies in the fact that the kernel of the maximal relation contains elements which are not functions with values in the initial space. In [5], [6], where generalized resolutions of symmetric relations generated by a formally self-adjoint expression and a nonnegative operator function were described in the infinite-dimensional case, it was initially supposed that a condition holds which guarantees that the kernel of the maximal relation contains no such elements.

In this paper, we consider a first order differential expression with unbounded operator coefficient generating a uniformly well-posed Cauchy problem. No conditions on the kernel of the maximal relation are imposed. In Theorem 1, we describe the restrictions of the maximal relation for which the inverse is an unbounded everywhere defined operator. We establish a criterion for the corresponding operator function to be holomorphic. In Theorem 2, we consider the situation when the operators inverse to restrictions of the maximal relation may be unbounded.

2. Let H be a separable Hilbert space with scalar product $(\cdot\,,\,\cdot)$ and norm $\|\cdot\|$, and let $A(t)$ be a strongly measurable on a finite interval $[0,b]$ operator function whose values are bounded self-adjoint operators in H such that $(A(t)x,x) \geq 0$ almost for all $t \in [0,b]$ and all $x \in H$. We assume that the norm $\|A(t)\|$ is summable on $[0,b]$.

On the set of continuous functions on a segment $[0,b]$ with values in H, we introduce the norm

$$\|y\|_p = \left(\int_0^b \|A^{1p}(t)y(t)\|^p dt \right)^{1/p}, \quad 1 < p < \infty.$$

Identifying with zero the functions y for which $\|y\|_p = 0$ and taking the completion, we obtain a Banach space. We denote this space by $\mathrm{B} = L_p(H, A(t); 0, b)$. Let $\widetilde{y}$ be an element of B, i. e., a class of functions identified with respect to the norm $\|\cdot\|_p$. For the sake of simplicity, we will say that a function $y(t)$ from the class $\widetilde{y}$ belongs to B. The class of functions containing y will be denoted by $\widetilde{y}$.

Let $G(t)$ be the set of elements from H on which the operator $A(t)$ vanishes, $H(t)$ the orthogonal complement of $G(t)$ in H, $H = H(t) \oplus G(t)$, and let $A_0(t)$ be the restriction of $A(t)$ to $H(t)$. By $H_\tau(t)$ $(-\infty < \tau < \infty)$ we denote the Hilbert scale of subspaces generated by the operator $A_0^{-1}(t)$ ([7], p. 65). For a fixed t, the operator $A_0^\alpha(t)$ $(\alpha > 0)$ maps continuously and bijectively $H_\beta(t)$ onto $H_{\beta+\alpha}(t)$