Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 49, No. 12, pp. 71–74, 2005 *UDC 519.853*

CONVERGENCE OF THE GRADIENT PROJECTION METHOD FOR A CERTAIN CLASS OF NONCONVEX PROBLEMS IN MATHEMATICAL PROGRAMMING

Yu.A. Chernyaev

In this paper, we generalize the gradient projection method which is used for optimization of smooth functions to the case of nonconvex admissible sets. In [1], one introduces the notion of a preconvex set as a set, whose supplement to its convex shell is convex, and proves that this set is always representable as a set-theoretical difference of two convex sets. In [2]–[4], one generalizes the gradient projection method to the case of preconvex admissible sets with a nonempty interior. In this paper, we generalize the obtained results to the case of a set-theoretical difference of an arbitrary convex set and a union of several convex sets. We obtain the necessary extremum conditions and formulate several assertions about the convergence of the proposed method.

1. The problem definition and the algorithm

Consider the following problem: find a point which satisfies the necessary condition for a local minimum of a function $\varphi(x)$ on a set X in the n-dimensional Euclidean space R^n, where $\varphi(x) \in C^1(X)$, and X is a set-theoretical difference of certain sets F and $\bigcup_{i=1}^{l} \operatorname{int} G_i$; in addition, F and G_i are convex and closed, the sets of inner points of X and G_i, $i = \overline{1,l}$, are nonempty. Let each set G_i, $i = \overline{1,l}$, at any its boundary point x have a unique support hyperplane, whose normal $N^i(x)$ is considered to be outer, i. e., for all $y \in G_i$ we have $\langle N^i(x), y - x\rangle \le 0$. Assume that for each i the unit vector $n^i(x)$ of the normal $N^i(x)$ is a continuous vector-function at the boundary G_i, i. e., for any $\varepsilon > 0$ one can find $\delta(\varepsilon) > 0$ such that the condition $\|x - y\| < \delta(\varepsilon)$ which holds for arbitrary x and y, belonging to the boundary G_i, implies the inequality $\|n^i(x) - n^i(y)\| < \varepsilon$. The necessary condition for a local minimum of $\varphi(x)$ on the set X of the mentioned form is described below.

Let us introduce the following notation: $s^i(x)$ is a projection of a point x onto the set G_i, $n^i(x)$ is the unit vector of the normal of the support hyperplane to G_i at the point $s^i(x)$, $\Gamma^i(x) = \{e \in R^n : \langle n^i(x), e - s^i(x)\rangle \ge 0\}$, $P(x) = F \cap \Gamma^1(x) \cap \Gamma^2(x) \cap \cdots \cap \Gamma^l(x)$. Projections $s^i(x)$ are defined uniquely, because G_i, $i = \overline{1,l}$, are convex sets of the Euclidian space R^n. Since each G_i, $i = \overline{1,l}$, at any its boundary point x has only one support hyperplane, the vectors $n^i(x)$ and, consequently, the half-spaces $\Gamma^i(x)$, $i = 1, 2, \dots, l$, are defined uniquely for any $x \in X$. If for certain i the points x and $s^i(x)$ do not coincide, then the vectors $n^i(x)$ and $x - s^i(x)$ have the same direction, therefore $\langle n^i(x), x - s^i(x)\rangle > 0$, i. e., $x \in \operatorname{int} \Gamma^i(x)$. If x and $s^i(x)$ coincide, then x belongs to the boundary of $\Gamma^i(x)$. Since $x \in X \subset F$ and for each $i = 1, 2, \dots, l$, $x \in \Gamma^i(x)$, we have that always $x \in P(x)$.

We propose the following algorithm which constructs sequential approximations.

Step 0. Put $k = 0$.

Step 1. Let $x_k \in X$ be the k-th approximation.

Step 2. Define the points $s^i(x_k)$, $i = \overline{1,l}$.

Step 3. Construct the half-spaces $\Gamma^i(x_k)$, $i = \overline{1,l}$.