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THE EFFICIENT CHOICE OF DISCRETE DATA FOR SOLUTION OF ILL-POSED PROBLEMS

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1. Problem definition

Following the investigation of papers [1]–[3], consider the problem of efficient discretization of operator equations of the I kind

$$Ax = f, \quad f \in \operatorname{Range}(A). \tag{1}$$

The problem is to construct the algorithms which guarantee the given level of accuracy for solutions of (1) at the minimal possible expense of computational resources. In what follows, we understand these resources as discrete data presented as values of scalar products calculated for the operator and the right-hand side of the equation under consideration.

Assume that in formula (1) the operator A belongs to the totality $\mathcal{C}(X)$ of compact linear operators which act in a real Hilbert space X. Assume that in X one has defined the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|x\| = \sqrt{(x,x)}$. Assume that instead of the exact right-hand side f we know only some perturbation $f_\delta \in X$ such that $\|f - f_\delta\| \le \delta$ and $f_\delta \notin \operatorname{Range}(A)$. We will approximate the normal solution $x^\dagger$ of equation (1), i. e., the solution of (1) with the minimal norm in X.

Let us now define the class of equations (1) under consideration. Concerning $x^\dagger$, assume that for some $\nu > 0$ and $\rho \ge 1$ it belongs to the set

$$\mathcal{M}_{\nu,\rho}(A) = \{x : x = |A|^\nu u,\ \|u\| \le \rho\}, \quad |A| = (A^*A)^{1/2}, \tag{2}$$

where the operator A^* is adjoint to A. Denote by $\mathcal{H}^r$, $r = 1, 2, \dots$, the class of operators $A \in \mathcal{C}(X)$, $\|A\| \le 1$, such that for any $m = 1, 2, \dots$ we have

$$\|(I - P_m)A\| \le m^{-r}, \quad \|A(I - P_m)\| \le m^{-r}. \tag{3}$$

Here I is the identity operator in X, and P_m is an orthoprojector onto the linear shell of the first m elements of a certain orthonormal basis $\{e_k\}_{k=1}^\infty$ in X, i. e., $P_m = \sum\limits_{k=1}^{m} e_k(\cdot, e_k)$.

An example of equation (1), whose solution belongs to set (2) and the operator does to $\mathcal{H}^r$, is the integral Fredholm equation

$$\widetilde{A}x(t) := \int_0^1 h(t,\tau)x(\tau)\,d\tau = f(t)$$