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CONVERGENCE OF THE CONDITIONAL GRADIENT METHOD

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We study the convergence of the conditional gradient method (e. g., [1]–[4]) in the problem of approximate minimization of functionals defined on bounded closed subsets of a reflexive space. We establish sufficient convergence conditions applicable, in particular, to nonconvex functionals. We describe applications to the optimal control problems.

Let us introduce the following basic notation: X and X^* are, respectively, a real Banach space and its conjugate one, $\langle x, x^* \rangle$ is the value of a functional x^* from X^* at an element x from X; $\mathrm{Cv}(X)$ is the set of all nonempty convex closed subsets of the space X. We call a mapping $F : M_1 \to M_2$ (M_1, M_2 are subsets of Banach spaces X_1, X_2, respectively) bounded, if for each bounded in X_1 set $M \subset M_1$ the range of $F(M)$ is a bounded subset of X_2.

Assume that the used below functionals are real. If $f : M \to \mathbb{R}$ is a functional on the set M, then $\arg\min\limits_M f$ is the set of points of the global minimum of f on M: the inclusion $x_* \in \arg\min\limits_M f$ is equivalent to the inequality $f(x_*) \le f(x) \quad \forall x \in M$. We understand the derivative $f'(x_0)$ of a functional f defined on a certain neighborhood $O(x_0)$ of a point x_0 of a Banach space X in a standard way ([1], pp. 18–19). Denote the totality of continuously differentiable on a set $U \subset X$ functionals by $C^1(U)$; $C^{1,\nu}(U)$ $(0 < \nu \le 1)$ is a part of $C^1(U)$ which consists of functionals f, whose derivatives satisfy the inequality $\|f'(u) - f'(v)\|_{X^*} \le K\|u - v\|_X^\nu$; the constant K is independent of u, v from U.

1. In what follows, X is a separable reflexive Banach space, $\|\cdot\|$ and $\|\cdot\|_*$ are the norms in X and X^*, respectively, the symbols $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively, $Q \in \mathrm{Cv}(X)$. Denote by $S(Q)$ the class of bounded mappings $F : Q \to X^*$ which satisfy the following condition: an arbitrary sequence $x_n \in X$ such that

$$x_n \rightharpoonup x, \quad F(x_n) \rightharpoonup x^*, \quad \varlimsup_{n\to\infty} \langle x_n, F(x_n)\rangle \le \langle x, x^*\rangle,$$

satisfies the correlations $x_n \to x$, $x^* = F(x)$. The close to $S(Q)$ classes of operators were studied by many authors (see, e. g., [5], [6] and references therein). Note that the contraction of an operator F from the class $S(X)$ onto a set Q belongs to the class $S(Q)$.

We call an element $x \in Q$ a singular point of a mapping $F : Q \to X^*$, if $\langle v - x, F(x)\rangle \ge 0$ $\forall v \in Q$. Unless the contrary is indicated, below we assume that Q is a bounded subset of X. In this case the set $\mathfrak{K}(F)$ of all singular points of the mapping F from the class $S(Q)$ is a compact [6], [7]. For any x from Q the linear functional $z \to \langle z, F(x)\rangle$ attains its minimum on the set Q; the function $\eta(x) = \min\{\langle z - x, F(x)\rangle,\ z \in Q\}$ makes sense. Evidently, $\eta(x) \le 0 \quad \forall x \in Q$, and $\{x \in Q,\ \eta(x) = 0\} = \mathfrak{K}(F)$.

We call a sequence Q_n $(n = 0, 1, \dots)$ of sets of the class $\mathrm{Cv}(X)$ exhausting a set Q, if $Q_n \subset Q_{n+1}$ $(n = 0, 1, \dots)$ and the union of all sets Q_n is everywhere dense in the set Q. Put $\eta_n(x) = \min\{\langle z - x, F(x)\rangle,\ z \in Q_{n+1}\}$ $(n = 0, 1, \dots, x \in Q_n)$.