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NILPOTENCY OF THE ENGEL ALGEBRAS

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In [1] it was proved that a finitely generated Lie algebra over a field of characteristic p, which satisfies the n-th Engel condition for $n < p$, is nilpotent. This result stimulated a solution of a similar problem for both the Malcev algebras and the binary Lie algebras, which generalize in a natural way the concept of the Lie algebras. In [2], an analog of the Engel theorem was proved for the finite-dimensional Malcev algebras. This result was sharpened in [3]. It was shown that a finitely generated Malcev algebra satisfying the n-th Engel condition is nilpotent if either $p \geq n$, or $p = 0$. Previously, in [4], it was established that the Engel binary Lie algebras which satisfy the maximality condition for subalgebras are nilpotent. In this article we solve the problem of nilpotency of the finite-dimensional Engel algebras with anticommutative multiplication.

Definition 1. An algebra G with an anticommutative multiplication is called an *Engel algebra* if $\forall x \in G$ the operator $R_x : g \to g \circ x$, $g \in G$, is nilpotent.

Definition 2. An algebra G is said to be *nilpotent* if a natural number n exists such that the product of any n elements of G for any arrangement of brackets equals zero.

Definition 3. An algebra G is said to be *right-nilpotent* if a natural number n exists such that $R_{a_1} R_{a_2} \dots R_{a_{n-1}} a_n = 0$ for any elements $a_1, \dots, a_n$ of G.

Let us define inductively a series of subspaces in G:

$$G^1 = G, \quad G^n = G^{n-1} \circ G + G^{n-2} \circ G^2 + \dots + G \circ G^{n-1}, \quad G^{(1)} = G, \quad G^{(n)} = G^{(n-1)} \circ G.$$

The chain of subspaces

$$G^1 \supseteq G^2 \supseteq \dots \supseteq G^n \supseteq \dots$$

is a chain of ideals of G, and the chain

$$G^{(1)} \supseteq G^{(2)} \supseteq \dots \supseteq G^{(n)} \supseteq \dots$$

is a chain of right-side ideals. Given that G is anticommutative, the latter chain consists of its ideals. In terms of these subspaces the above definitions can be formulated as follows: An algebra G is nilpotent if $G^n = 0$ for a certain n, and G is right-nilpotent if $G^{(n)} = 0$.

In the case of an anticommutative algebra G, one can easily verify that (see [5], p. 102) $G^{2^n} \subseteq G^{(n)}$. Consequently, in this case, both the notions of nilpotency and right nilpotency coincide.

Theorem 1. *Let G be an Engel algebra over a field K. If $\dim_K G \leq 4$, then G is nilpotent.*

We subdivide the proof into a series of lemmas.

Lemma 1. *In an Engel algebra G, $\dim_K G \geq 2$, a two-dimensional subalgebra with trivial multiplication exists.*