Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 45, No. 11, pp.12–14, 2001 *UDC 517.95*

THE RIEMANN METHOD OF THE GOURSAT PROBLEM FOR A CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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In [1]–[3] the solution of the Goursat problem for third order equations in the three-dimensional space was expressed via a large number of addends, which makes difficult the usage of this solution in the study of other boundary value problems. In this article we obtain a compact form of this solution.

Consider the equation

$$\mathcal{L}(\mathcal{U}) \equiv \mathcal{U}_{xyz} + a_1\mathcal{U}_{xy} + a_2\mathcal{U}_{xz} + a_3\mathcal{U}_{yz} + b_1\mathcal{U}_x + b_2\mathcal{U}_y + b_3\mathcal{U}_z + c\mathcal{U} = 0 \tag{1}$$

in the first octant $G = \{(x,y,z) : 0 < x,y,z < +\infty\}$. We assume that $a_{1xy}, a_{2xz}, a_{3yz}, b_{1x}, b_{2y}, b_{3z}, c \in \mathbb{C}(G)$.

Goursat problem. *Find the solution of equation* (1) *in the domain* G, *which is continuous in* $\overline{G}$ *and satisfies the boundary value conditions*

$$\mathcal{U}(0,y,z) = \varphi_1(y,z), \quad 0 \le y,z < +\infty, \tag{2}$$

$$\mathcal{U}(x,0,z) = 0, \quad 0 \le x,z < +\infty, \tag{3}$$

$$\mathcal{U}(x,y,0) = 0, \quad 0 \le x,y < +\infty. \tag{4}$$

Theorem. *If* $\varphi_{1ts}(t,s) \in \mathbb{C}[0,+\infty;\ 0,+\infty]$, *then the unique solution of the Goursat problem is defined by the formula*

$$\mathcal{U}(x,y,z) = \int_0^y dt \int_0^z [\varphi_{1ts}(t,s) + a_1(0,t,s)\varphi_{1t}(t+s) + \\ + a_2(0,t,s)\varphi_{1s}(t,s) + b_1(0,t,s)\varphi_1(t,s)]R(0,t,s;x,y,z)ds. \tag{5}$$

Proof. We will prove the validity of this proposition by the Riemann method. The Riemann function for equation (1) exists (see [4]) and is uniquely defined. By the straightforward differentiation one can easily prove the identity

$$R\mathcal{L}(\mathcal{U}) - \mathcal{U}\mathcal{L}^*(R) = \frac{1}{6}(P_x + Q_y + H_z),$$

where $\mathcal{L}^*(R)$ is the operator conjugate to the operator $\mathcal{L}(\mathcal{U})$,

$$P = 2(\mathcal{U}R)_{yz} - 3[(R_z - a_1R)\mathcal{U}]_y - 3[(R_y - a_2R)\mathcal{U}]_z + 6[R_{yz} - (a_1R)_y - (a_2R)_z + b_1R]\mathcal{U},$$
$$Q = 2(\mathcal{U}R)_{xz} - 3[(R_z - a_1R)\mathcal{U}]_x - 3[(R_x - a_3R)\mathcal{U}]_z + 6[R_{xy} - (a_1R)_x - (a_3R)_z + b_2R]\mathcal{U},$$
$$H = 2(\mathcal{U}R)_{xy} - 3[(R_y - a_2R)\mathcal{U}]_x - 3[(R_x - a_3R)\mathcal{U}]_y + 6[R_{xy} - (a_2R)_x - (a_3R)_y + b_3R]\mathcal{U}.$$