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ON NONINTEGER VERTICES OF THE POLYTOPE OF MULTI-INDEX AXIAL CHOICE PROBLEM

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1. Introduction

In contrast to the classical transport problem (TP) the polytope (or polyhedron) of the multi-index TP may have noninteger vertices (tops) for noninteger input data (see, e. g., [1]–[3]). Naturally, this makes difficult the search of integer solutions, turning, for instance, the problem of checking of integer compatibility of 3-index planar TP into an NP-complete problem (see [4]). The NP-completeness of the axial and planar 3-index choice problem was established in [5], [6]. In particular, this explains the profound interest to properties and methods of solving multi-index TP, which justifies investigations concerning the structure of polytopes of this class of problems.

In the present article we investigate noninteger vertices of the polytope of the multi-index axial choice problem, the latter being of ample use in applications.

2. Basic definitions, properties, and results

Let natural numbers p and n satisfy the inequalities $p \geq 2$ and $n \geq 2$. Consider the polytope $M(p,n)$ given via the conditions

$$\sum_{i_1=1}^{n} \cdots \sum_{i_{s-1}=1}^{n} \sum_{i_{s+1}=1}^{n} \cdots \sum_{i_p=1}^{n} x_{i_1 \ldots i_p} = 1 \quad \forall i_s \in N_n, \ \forall s \in N_p, \tag{1}$$

$$x_{i_1 \ldots i_p} \geq 0 \quad \forall (i_1, \ldots, i_p) \in N_n^p, \tag{2}$$

where $N_t = \{1, 2, \ldots, t\}$, $N_n^p = \underbrace{N_n \times \cdots \times N_n}_{p}$. Since the integer points of the polytope $M(p,n)$ serve as the admissible domain of a p-index axial choice problem (see [1], p. 308), the polytope $M(p,n)$ will be termed here the polytope of the p-index axial choice problem. Note that with $p = 2$ this polytope is the polytope of the well-known and well-studied assignment problem.

We shall keep the terminology and notation used in [1].

The following basic properties of the polytope $M(p,n)$ are known (see [1], [2]):

1) the rank of the matrix of system (1) equals $(n-1)p+1$; this property implies that a vertex of the polytope $M(p,n)$ contains at most $(n-1)p+1$ positive components;

2) the dimension of the polytope $M(p,n)$ equals $n^p - (n-1)p - 1$;

3) the quantity of integer vertices of the polytope $M(p,n)$ equals $(n!)^{p-1}$;

4) the polytope $M(2,n)$ is integer, i. e., all its vertices have integer components;

5) the diameter of the polytope $M(2,n)$ for $n \geq 4$ equals 2;

6) to each vertex of the polytope $M(2,n)$ $2^{n-1}n^{n-2}$ admissible bases correspond;

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