Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 43, No. 9, pp.69–73, 1999 *UDC 512*

GENERALIZED EXTERIOR ALGEBRAS AND THEIR APPLICATIONS

M.P. Burlakov

1. The Grassmann and Clifford exterior algebras have long been a popular tool in various areas of both the mathematics and theoretical physics. The basis for all applications of the exterior algebras is the identity

$$\mathbf{v} \wedge \mathbf{v} = \mathbf{0}, \tag{1}$$

where $\mathbf{v}$ is a vector of the underlying vector space of a Grassmann algebra. In the present article we consider generalizations of the exterior algebras in which the identity holds

$$\mathbf{v}^m \equiv \underbrace{\mathbf{v} \cdot \mathbf{v} \cdots \mathbf{v}}_{m} = \mathbf{0}. \tag{2}$$

Clearly, identity (1) is a particular case of (2) for $m = 2$, and this generalization can be used for modifications of various applications of the exterior algebras.

It should be noted that the existence of exterior algebras with characteristic identity (2) is by no means evident. There exists a simple argument passing from one monograph to another which asserts that an exterior algebra is necessarily an algebra with anticommutative multiplication law. We shall cite this argument here in order that the reader can assess the specificity of structure of generalized exterior algebras as well as "subjective reasons" why the exterior algebras with defining identity (2) did not attract attention of mathematicians.

Let $\mathbf{E}_n$ be a vector space with a basis $\mathbf{e}_i$ satisfying the commutation rules of the following general form

$$\mathbf{e}_i \cdot \mathbf{e}_j = \alpha \mathbf{e}_j \cdot \mathbf{e}_i. \tag{3}$$

Then

$$\mathbf{e}_i \cdot \mathbf{e}_j = \alpha\, \mathbf{e}_j \cdot \mathbf{e}_i = \alpha^2\, \mathbf{e}_i \cdot \mathbf{e}_j,$$

i.e., $\alpha^2 = 1$ and, consequently, either $\alpha = 1$, or $\alpha = -1$. If $\alpha = 1$, then we obtain a commutative algebra over $\mathbf{E}_n$ isomorphic to the algebra of polynomials in n variables, and if $\alpha = -1$, then we obtain the Grassmann exterior algebra.

2. At a first glance this simple argument excludes the possibility of existence of exterior algebras with defining identities of the form (3). However, we can avoid this obstruction by requiring that rules (3) take place for $i > j$. We assume that $\mathbf{E}_n$ is a complex vector space and set

$$\begin{gathered} \mathbf{e}_i \cdot \mathbf{e}_j = \alpha\, \mathbf{e}_j \cdot \mathbf{e}_i, \quad i > j; \\ \mathbf{e}_i^m = 0, \end{gathered} \tag{4}$$

where α is a primitive m-th root of the unity.