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PROLONGATIONS OF TENSOR FIELDS AND CONNECTIONS TO THE WEIL BUNDLES

A.Ya. Sultanov

Bundles of $\mathbb{A}$-points over a differentiable manifold M_n were introduced by A. Weil (see [1]). A. Morimoto in [2] constructed lifts of tensor fields of type (p, q) $(p, q = 0$ or $1)$ from M_n to the Weil bundle $M_n^{\mathbb{A}}$ for an arbitrary local algebra $\mathbb{A}$. In the same paper he constructed the complete lift ∇^C of a linear connection ∇ on M_n. Manifolds over local algebras and connections on such manifolds were studied in [3]–[5].

In the present article we construct lifts of functions and tensor fields from M_n to $M_n^{\mathbb{A}}$ and synectic lifts in the sense of A.P. Shirokov of a linear connection. It is established that every synectic lift of a linear connection is the realization of a holomorphic connection on $M_n^{\mathbb{A}}$.

1. The space of $\mathbb{A}$-points over $\mathbb{R}^n$

Let $C^\infty(\mathbb{R}^n)$ be the algebra of C^∞ functions on $\mathbb{R}^n$, $\mathbb{A}$ a local algebra. Recall that a local algebra over $\mathbb{R}$ is a commutative associative unitary linear algebra $\mathbb{A}$ which possesses a nilpotent ideal $\mathbb{I}$ such that the quotient algebra $\mathbb{A}/\mathbb{I}$ is isomorphic to the field of real numbers $\mathbb{R}$ (see [6], p. 146).

The least natural number r such that $\mathbb{I}^{r+1} = 0$ is called the height of $\mathbb{A}$. The dimension of the quotient algebra $\mathbb{I}/\mathbb{I}^2$ is called the width of $\mathbb{A}$. Regarded as a vector space, $\mathbb{A}$ is the direct sum $\mathbb{R} \oplus \mathbb{I}$.

Definition 1.1 (see [7]). An $\mathbb{A}$-point near to $x \in \mathbb{R}^n$ is a homomorphism $x' : C^\infty(\mathbb{R}^n) \to \mathbb{A}$ satisfying the condition $x'(f) \equiv f(x) \bmod \mathbb{I}$ for any function $f \in C^\infty(\mathbb{R}^n)$.

We denote by $(\mathbb{R}^n)^{\mathbb{A}}_x$ the set of $\mathbb{A}$-points x' near to $x \in \mathbb{R}^n$ and put $(\mathbb{R}^n)^{\mathbb{A}} = \bigcup\limits_{x \in \mathbb{R}^n} (\mathbb{R}^n)^{\mathbb{A}}_x$. From the definition of an $\mathbb{A}$-point and the properties of the algebra $\mathbb{A}$ it follows $x'(c) = c$ for any constant function $c \in C^\infty(\mathbb{R}^n)$.

In what follows we shall need a local representation of a homomorphism $x' \in (\mathbb{R}^n)^{\mathbb{A}}$. Let $f \in C^\infty(\mathbb{R}^n)$ and $x_0 = (x_0^1, x_0^2, \dots, x_0^n) \in \mathbb{R}^n$. A neighborhood of x_0 exists in which f is expressed as follows

$$f = f(x_0) + \sum_{|p|=1}^{r} \frac{1}{p!} D_p f(x_0)(x - x_0)^p + \sum_{|p|=r+1} \frac{1}{p!} (D_p f \circ \xi)(x - x_0)^p. \tag{1.1}$$

Here $p = (p_1, p_2, \dots, p_n)$ is a multi-index, p_i are nonnegative integers, $|p| = p_1 + p_2 + \dots + p_n$, $p! = p_1! p_2! \dots p_n!$, $(x - x_0)^p = (x^1 - x_0^1)^{p_1} (x^2 - x_0^2)^{p_2} \dots (x^n - x_0^n)^{p_n}$, x^i are the coordinate functions, $\xi = x + \theta(x - x_0)$, $0 < \theta < 1$, and

$$D_p f(x_0) = \frac{\partial^{|p|} f}{(\partial x^1)^{p_1} \dots (\partial x^n)^{p_n}} (x_0).$$