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ON GEOMETRY OF PROJECTIVE SUBMERSIONS OF RIEMANNIAN MANIFOLDS

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Introduction

The projective (geodesic, in other terms) diffeomorphisms were investigated in many works (see [1], [2] about this). Among these works we should mention the works on the global geometry of projective diffeomorphisms of compact Riemannian manifolds (see, e. g., [3], [4]). In these papers it was discovered that the negativeness of sectional curvature is an obstruction to the existence of these diffeomorphisms; this fact can be considered as a main result. Developing the theory of projective mappings, a number of authors lifted the requirement that the dimensions of involved manifolds are to be equal and, what is more, the restriction on the rank of mapping (see [5]–[8]). At the same time, in the field of research the projective mappings of complete Riemannian manifolds were included (see [9], [10]).

The present article supplements these investigations. Here we shall describe geometry of a Riemannian manifold which admits a projective submersion on a Riemannian manifold of lesser dimension, and find an obstruction to the existence of a local projective submersion of Riemannian manifold and of a global projective submersion of compact Riemannian manifold with boundary.

1. Projective submersions with geodesically complete fibres

1.1. Let M and N be m- and n-dimensional Riemannian manifolds, respectively, with metrics g and g', and ∇ and ∇' be the corresponding Levi-Civita connections. Let us consider a smooth mapping $f : M \to N$ of class C^∞. We denote by $f^{-1}TN$ the vector bundle with the base M whose fibre over a point $x \in M$ is $T_{f(x)}N$. The differential $f_* : TM \to TN$ is a linear mapping, i. e., $f_{*x} \in T_{f(x)}N \otimes T^*M$ for each x in M. Therefore f_* can be considered as a vector-valued 1-form on M with values in the vector bundle $f^{-1}TN$.

The mapping $f : M \to N$ is called *a mapping of constant rank* if $r = \mathrm{Rang}(f_{*x})$ does not depend on a choice of $x \in M$. In part, if $\mathrm{Rang}(f_{*x}) = \dim N$ at each point $x \in M$, then f is called *a submersion*.

1.2. Let $\gamma : t \in J \to \gamma(t) \in M$ be a pregeodesic of a manifold M with linear connection ∇, then $\nabla_X X = \varphi(t)X$, where $X = d\gamma/dt$ is the vector field tangent to γ. Let us change the parameter along γ so that t becomes *an affine parameter*. Then $\nabla_X X = 0$, and γ is called *a geodesic line*. By analyzing the last equation, one can conclude (see [11], p. 183) that either γ is an immersion, i. e., $d\gamma/dt \neq 0$ for all $t \in J$, or $\gamma(J)$ is a point of M.

If $f : M \to N$ sends each pregeodesic line γ of M into a pregeodesic line $\gamma' = f(\gamma)$ of N, then the mapping is said to be *projective* (see, e. g., [12], [13]). Moreover, if f is a mapping of constant rank $r < m$, then each pregeodesic line γ which is an integral curve of the distribution $\ker f_*$, is