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# ON SPECIAL CONNECTIONS WHICH DETERMINE REPRESENTATIONS OF ZERO CURVATURE FOR SECOND ORDER EVOLUTION EQUATIONS

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## Introduction

The concept of a connection determining a representation of zero curvature for a given partial differential equation is of primary importance in the mathematical theory of solitons. The connections determining representations of zero curvature allow to give geometric interpretations of the inverse problem equations, Bäcklund transformations, and pseudopotentials of Wahlquist and Estabrook (see [1], [2]). The first example of such a connection was that by R. Hermann in [3], associated with the prolongation structure by H.D. Wahlquist and F.B. Estabrook (see [4]). Seemingly, one can assert that [3] gave rise to a new chapter in the differential geometry.

The question on the existence of a connection determining a representation of zero curvature for a given partial differential equation is among the unsolved fundamental problems. In the present article we prove (see § 3, Theorem 2) that for an arbitrary second order evolution equation (with one spatial variable) this problem has a solution (in this connection, see also Remark 3).

We study connections of a special type (see § 2), which are termed here $(\rho, \rho_1)$-connections. The curvature tensor of such a connection contains a subtensor with the two components $\rho$ and $\rho_1$, where $\rho$ is a relative invariant. In the case where the tensor with components $\rho$ and $\rho_1$ vanishes on and only on solutions (more exactly, on the corresponding lifts of solutions) of a given evolution equation, the $(\rho, \rho_1)$-connection determines a representation of zero curvature for this equation. We prove a theorem on existence of a $(\rho, \rho_1)$-connection for a given tensor with components $\rho$ and $\rho_1$. From this theorem it follows that (see § 3) for any second order evolution equation a connection exists determining a representation of zero curvature.

All the constructions in the present article are local in character.

We use systematically the Cartan–Laptev invariant analytic method and, in particular, the theory of structure forms on fiber bundles constructed in [5]–[9]. A detailed exposition of the Cartan–Laptev method can be found in [10]. Monograph [11] is devoted to further development of this method.

## 1. Connections determining a representation of zero curvature and their general properties

1.1. Consider the second order partial differential equation

$$F(t, x^1, \dots, x^n, u, u_{\hat{j}}, u_{\hat{k}\,\hat{l}}) = 0, \qquad (1)$$

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