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ON THE GEOMETRY OF TOTALLY GEODESIC RIEMANNIAN FOLIATIONS

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Let M be a smooth connected reducible Riemannian manifold. Then on M two parallel foliations exist, each of them being the orthogonal complement to the other (see [1], p. 173). If M is complete and simply connected, then the de Rham theorem is valid, which states that M is isometric to the direct product of any leaf of one foliation and any leaf of the other foliation (ibid., p. 180). In this case, both foliations are simultaneously Riemannian and totally geodesic. In the present article we study a totally geodesic Riemannian foliation on M without the assumption that the orthogonal complement is integrable. In what follows the smoothness means the differentiability of class C^∞.

1. Let M be a smooth connected complete Riemannian manifold of dimension n with the Riemannian metric g, ∇ the Levi-Civita connection, and F a smooth foliation of dimension k on M (see [2]; [3], p. 24). For a point $p \in M$, we denote by $L(p)$ the leaf of F passing through p, by $F(p)$ the tangent space of $L(p)$ at p, and by $H(p)$ the orthogonal complement to $F(p)$ at T_pM. Two subbundles of the tangent bundle (smooth distributions) arise: $TF = \{F(p) : p \in M\}$ and $H = \{H(p) : p \in M\}$ such that $TM = TF \oplus H$, where H is the orthogonal complement to TF. By the definition of a foliation, for every point $p \in M$ a neighborhood U of p and a local coordinate system $(x^1, x^2, \dots, x^k, y^1, y^2, \dots, y^{n-k})$ on U exist such that $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^k}\}$ is a basis for smooth sections of $TF|_U$ (the restriction of TF to U) (see [2]). Let $\omega^1, \omega^2, \dots, \omega^k$ be smooth differential I-forms, $\nu_1, \nu_2, \dots, \nu_m$ smooth vector fields on U such that $\{\omega^i, dy^\alpha\}$ form a local basis for sections of cotangent bundle, $\{\frac{\partial}{\partial x^i}, \nu_\alpha\}$ form a basis for sections of the tangent bundle $TU = TF|_U \oplus H|_U$ which is dual to $\{\omega^i, dy^\alpha\}$, where $m = n - k$, $1 \le i \le k$, $1 \le \alpha \le m$. Such a neighborhood will be denoted by $U(x, y)$.

Now suppose that F is a Riemannian foliation with respect to g (see [2], [3]). The latter means that in each $U(x, y)$ the Riemannian metric g has the form

$$g_{i_j}(x, y)\omega^i\omega^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta,$$

where $1 \le i,\, j \le k$, $1 \le \alpha,\, \beta \le m$, $x = (x^1, x^2, \dots, x^k)$, $y = (y^1, y^2, \dots, y^m)$.

Remark. To simplify the notation, in the expressions with summation over repeated indices we shall drop the summation sign.

A piecewise smooth curve $\gamma : [0, 1] \to M$ will be called horizontal if $\dot\gamma(t) \in H(\gamma(t))$ for every $t \in [0, 1]$. A piecewise curve lying in a leaf of F is said to be vertical.

We set $I = [0, 1]$. Let $\nu : I \to M$ be a vertical curve, and $h : I \to M$ a horizontal curve with $h(0) = \nu(0)$. A piecewise smooth mapping $P : I \times I \to M$ such that 1) $t \to P(t, s)$ is a vertical curve for each $s \in I$; 2) $s \to P(t, s)$ is horizontal curve for each $t \in I$; 3) $P(t, 0) = \nu(t)$ for $t \in I$, $P(0, s) = h(s)$ for $s \in I$, is called a vertical-horizontal homotopy (see [4], [5]). If for each pair of vertical and horizontal curves $v, h : I \to M$ with $v(0) = h(0)$ both the corresponding vertical-horizontal homotopy P exists, then the distribution H is called an Ehresmann connection