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INTERMEDIATE INTEGRALS OF THE MONGE–AMPÈRE EQUATIONS

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Intermediate integrals provide one of the ways to find exact solutions of differential equations. In the present article the intermediate integrals for the Monge–Ampère equations are described. The choice of the class of equations stipulates the fact that our results are obtained in the framework of the geometry of 1-jets. We give a criterion of existence of the intermediate integrals for the Monge–Ampère equations. Namely, the existence of intermediate integrals for a Monge–Ampère equation on an n-dimensional smooth manifold is equivalent to the existence of a symplectically exact characteristic 1-form for the second order linear differential operator associated with the same effective n-form as the Monge–Ampère operator is associated with. In the 2-dimensional case, this condition is equivalent to the existence of an eigenvector in the space of symplectically exact 1-forms for the operator generated by the two 2-forms: The differential of Cartan's 1-form and the effective form associated with the Monge–Ampère equation. For the Monge–Ampère equations associated with closed forms of constant class, we give a complete description of the intermediate integrals. Namely, if the Pfaffian of an equation of that kind is not a constant, then all the intermediate integrals are some functions of the Pfaffian. If the Pfaffian is a constant, then the Monge–Ampère equation is either hyperbolic, or parabolic. In the first of cases the equation determines two three-dimensional integrable distributions on the manifold of 1-jets, which intersect the Cartan distribution by nonintegrable distributions, and the intermediate integrals are the first integrals of any of these distributions. In the second case, the equation determines one 3-dimensional integrable distribution on the manifold of 1-jets, whose intersection with the Cartan distribution is an integrable distribution, and the intermediate integrals are the first integrals of this distribution. To illustrate the results obtained we give the complete description of the intermediate integrals for the Minkowski and Aleksandrov problems in the classical differential geometry.

We should note that the present article contains the complete exposition of the results announced in [1].

1. A family of the Lie structures on the manifold of 1-jets

Let M be a smooth manifold, $\dim M = n$, and μ_x the ideal of the ring $C^\infty(M)$ related to a point $x \in M : \mu_x = \{f \in C^\infty(M) \mid f(x) = 0\}$. The smooth vector bundle $\pi_k : J^k M \to M$ with fiber $J^k_x M = C^\infty(M)/\mu_x^{k+1} C^\infty(M)$ over $x \in M$ is called the bundle of k-jets. For any $k > l \geq 0$, a natural projection $\pi_{k,l} : J^k M \to J^l M$ exists. The image of a function $f \in C^\infty(M)$ in a fiber $J^k_x M$ is denoted by $j_k(f)_x = [f]^k_x$. Let $\mathcal{J}^k(M)$ stand for the module of smooth sections of the bundle $J^k M$, and $S_{j_k(f)} \subset \mathcal{J}^k(M)$ be the section defined by the relation $S_{j_k(f)}(m) = j_k(f)_m$, $m \in M$.

A smooth mapping $F : M_1 \to M_2$ generates the module homomorphism

$$\mathcal{J}^k(F) : \mathcal{J}^k(M_2) \to \mathcal{J}^k(M_1),$$
$$[f]^k_{m_2} \mapsto [F^*(f)]^k_{m_1},$$