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PSEUDO-LIOUVILLE SURFACES

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In 1846 J. Liouville introduced (see [1]) an interesting class of Riemannian surfaces which were further called the *Liouville surfaces.* In an appropriate coordinate system u^1, u^2 the metric of a Liouville surface is of the form (see [2], p. 151)

$$ds^2 = [X(u^1) + Y(u^2)] \cdot [(du^1)^2 + (du^2)^2],$$

where $X(u^1)$ and $Y(u^2)$ are some functions of the indicated variables. The equations of geodesics of Liouville surfaces admit a (quadratic) first integral of the form $a_{ij} du^i du^j = \text{const}$ $(i, j = 1, 2)$, where $a_{11} = (X+Y)Y$, $a_{22} = -(X+Y)X$, $a_{12} = 0$. Moreover, in the case of the proper Riemannian surfaces (with a positive definite metric form), this property characterizes the Liouville surfaces (see [2], p. 192). The characteristic equation

$$|a_{ij} - \lambda g_{ij}| = 0, \tag{1}$$

where a_{ij} are the components of the tensor which determines a first integral of geodesics, has two real nonzero roots $\lambda_1 = Y$, $\lambda_2 = -X$.

A natural question arises: Whether surfaces different from the Liouville ones exist for which the equations of geodesics also admit a quadratic first integral? Obviously, the fundamental form of such a surface cannot be positive definite. In addition, characteristic equation (1) may have both complex conjugate roots and one but multiple real root, which cannot take place in the case of the Liouville surfaces. The present article is devoted to the solution of the above indicated problem. As a generalization of the results obtained, we construct the Weyl surfaces (see [3], p. 153) for which the equations of geodesics admit a quadratic fractional first integral (see [4]).

1. The characteristic equation of a surface

Let V_2 be a pseudo-Riemannian surface possessing, generally speaking, an indefinite metric form. In terms of local coordinates u^i $(i, j, k = 1, 2)$, the line element of V_2 is of the form $ds^2 = g_{ij}(u^k) du^i du^j$, where g_{ij} is the metric tensor. We suppose that this surface admits a nondegenerate net of lines (see [2], p. 65) determined by a (nondegenerate) tensor a_{ij}. Consider characteristic equation (1) of this net. First of all, let us elucidate the conditions under which this equation has complex conjugate roots. Note that these roots never equal zero since the net tensor is nondegenerate.

It can be easily seen that, if $\lambda_{1,2} = U(u^k) \pm iV(u^k)$, $i^2 = -1$, are roots of equation (1), then the functions U, V and the coordinates of the tensors g_{ij} and a_{ij} satisfy the following two conditions:

$$\begin{aligned} a_{11}a_{22} - a_{12}^2 &= (U^2 + V^2)(g_{11}g_{22} - g_{12}^2), \\ a_{11}g_{22} - 2a_{12}g_{12} + a_{22}g_{11} &= 2U(g_{11}g_{22} - g_{12}^2). \end{aligned} \tag{2}$$