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THE METHOD OF UPPER AND LOWER SOLUTIONS FOR EQUATIONS OF ELLIPTIC TYPE WITH DISCONTINUOUS NONLINEARITIES

V.N. Pavlenko and O.V. Ul'yanova

Introduction

We consider a problem of existence of strong solutions of boundary value problems of the form

$$Lu \equiv -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} + \sum_{j=1}^{n} b_j(x)u_{x_j} + c(x)u = g(x, u(x)), \tag{1}$$

$$Bu|_\Gamma = 0, \tag{2}$$

in a bounded domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$, with the boundary Γ of class $C_{2,\alpha}$, $\alpha \in (0,1]$ (see [1], p. 23), where a differential operator L is uniformly elliptic in Ω, and (2) is either a homogeneous boundary Dirichlet condition $u|_\Gamma = 0$, or the third boundary value condition $\frac{\partial u}{\partial n_L} + \sigma(x)u|_\Gamma = 0$, $\frac{\partial u}{\partial n_L} = \sum\limits_{i,j=1}^{n} a_{ij}(x)u_{x_i}\cos(n, x_j)$, n is the exterior normal to the boundary Γ, $\cos(n, x_j)$ are direction cosines of the normal n, the function $\sigma \in C_{1,\alpha}(\Gamma)$ (see [1], p. 23) is nonnegative on Γ. The coefficients a_{ij}, b_j, and c of the operator L are continuous by Hölder with the index α together with the partial derivatives $(a_{ij})_{x_j}$ on $\overline{\Omega}$, $a_{ij}|_\Gamma \in C_{1,\alpha}(\Gamma)$, the nonlinearity $g(x,u)$ equals the difference between superpositionally measurable functions $g_2(x,u)$ and $g_1(x,u)$, which do not decrease by the variable u. We do not suppose that $g(x,u)$ is continuous with respect to u. By a strong solution of problem (1)–(2) we shall call the function $u \in \mathbf{W}_q^2(\Omega)$, $q > 1$, which satisfies equation (1) almost everywhere on Ω, for which the trace of $Bu(x)$ on Γ equals zero.

We shall obtain propositions concerning existence of strong solutions of problem (1)–(2). The proofs are based on an abstract scheme of the method of upper and lower solutions from [2]. In the works by H. Amann and his disciples (see [3]–[7]) the basement of a method for investigations of boundary value problems for equations of elliptic and parabolic types with smooth nonlinearities was given; this method was based on the use of upper and lower solutions. C.A. Stuart modified this approach in application to parabolic equations of elliptic type with a discontinuous nonlinearity $g(x,u) \equiv g(u)$ in [8]. His results obtained further development in [9]–[11]. The technique of upper and lower solutions was applied to the study of the first boundary value problem for equations of parabolic type with a discontinuous nonlinearity in [12], [13], and also in [2] and [14]. The most general theorems on the existence of strong solutions of problem (1)–(2) were established in [15] by topological methods with use of upper and lower solutions. In the case of the boundary Dirichlet condition and a formally selfadjoint operator L with $c(x) \equiv 0$, in [16] and [17] by means of the variational method the existence of strong solutions of problem (1)–(2) under restrictions upon the discontinuities of the nonlinearity $g(x,u)$, which were weaker than in [15], was proved. Namely,

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