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ON EQUALING ZERO BY THE GROUP OF HOMOMORPHISMS OF ABELIAN GROUPS

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In the present article we solve the question on the conditions, under which the group of Abelian group homomorphisms $\mathrm{Hom}(A, C)$ equals zero if C is a periodic group or homogeneous separable torsion-free group (in particular, a torsion-free group of rank 1). For a periodic group C the answer to the question posed above has been completely obtained. In the case of torsion-free groups C we obtain the complete answer for the groups A, whose type of each nonzero element of their factor-groups with respect to periodic part is not less than the type of the group C (in particular, for homogeneous Abelian torsion-free groups A, whose type coincides with the type of the group C). From the results obtained in the present article the description of coslender Abelian groups follows, which was obtained in [1].

First, consider the general situation. Let C be an arbitrary Abelian group. We denote by $\mathfrak{A}_C$ the class of all Abelian groups A with the property $\mathrm{Hom}(A, C) = 0$.

Proposition 1. *The class* $\mathfrak{A}_C$ *is closed with respect to*: a) *factor-groups*; b) *extensions*; c) *direct sums*; d) *direct products with nonmeasurable set of components if* C *is a slender group*; e) *direct limits*; f) *tensor products by an arbitrary Abelian group.*

Proof. a) Let $A \in \mathfrak{A}_C$ and B be a subgroup of the group A. We have $\mathrm{Hom}(A, C) = 0$. If we suppose that $\varphi \in \mathrm{Hom}(A/B, C)$ exists, $\varphi \neq 0$, then by considering the homomorphism $\varphi\eta$, where η is the canonical epimorphism of A onto A/B, we obtain that $\varphi\eta \in \mathrm{Hom}(A, C)$ and $\varphi\eta \neq 0$. We have arrived at a contradiction.

b) Suppose that in the exact sequence $0 \to B \to A \to D \to 0$ the groups B and D are taken from the class $\mathfrak{A}_C$. Let $f : A \to C$ be a homomorphism. We identify the group D and the group A/B. Construct the homomorphism $\psi : A/B \to C$ as follows: $\psi(a + B) = f(a)$. The condition $\mathrm{Hom}(B, C) = 0$ guarantees the correctness of the mapping constructed above. We have $\varphi\eta = f$ (η being the canonical epimorphism of A to A/B). Since $A/B \in \mathfrak{A}_C$, we have $\psi = 0$ and therefore $f = 0$. Consequently, $A \in \mathfrak{A}_C$.

c) Let $\{A_i\}_{i \in I}$ be a certain family of groups from the class $\mathfrak{A}_C$. Then $\mathrm{Hom}\big(\bigoplus_{i\in I} A_i, C\big) \cong \prod_{i\in I} \mathrm{Hom}(A_i, C) = 0$. Consequently, $\bigoplus_{i\in I} A_i \in \mathfrak{A}_C$.

d) Let $\{A_i\}_{i \in I}$ be a certain family of groups from the class $\mathfrak{A}_C$, where I is the set of nonmeasurable cardinality, and C is a slender group. Then, with regard for corollary 7 from [2], we have $\mathrm{Hom}\Big(\prod_{i\in I} A_i, C\Big) \cong \bigoplus_{i\in I} \mathrm{Hom}(A_i, C) = 0$. Hence, $\prod_{i\in I} A_i \in \mathfrak{A}_C$.

e) Let $\{A_i\ (i \in I);\ \pi_i^j\}$ be the direct spectrum of Abelian groups and homomorphisms (see [3], p. 68), where $A_i \in \mathfrak{A}_C$. If B is a subgroup generated by all elements of the form $a_i - \pi_i^j a_i$ $(i \le j)$, then $\varinjlim_I A_i = \bigoplus_{i\in I} A_i/B$ and by virtue of c) and a) we have $\varinjlim_I A_i \in \mathfrak{A}_C$.