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ON THE FUNCTIONS WHICH ARE THE SECOND MODULUS OF CONTINUITY

V.E. Geĭt

Let a function f belong to $C_{2\pi}$, i.e., f is 2π−periodic and continuous on the axis $(-\infty,\infty)$. For $0\le\delta\le\pi$, we set, as usual (see, e.g., [1], p. 115),

$$\omega_2(f,\delta)=\sup_{|h|\le\delta}\max_x|\Delta_h^2 f(x)|,$$

where the second difference with the step h is

$$\Delta_h^2 f(x)=f(x+2h)-2f(x+h)+f(x).$$

A function $\varphi(\delta)$ on $[0,\pi]$ is called a modulus of continuity of the second order if for a certain function $f_\varphi\in C_{2\pi}$ we have

$$\omega_2(f_\varphi,\delta)=\varphi(\delta),\quad 0\le\delta\le\pi.$$

For instance, all moduli of continuity are the moduli of continuity of the second order [1].

There is still no characteristic property of the second order continuity moduli. The problem of finding sufficient conditions is also of interest.

Theorem 1. *A function $\varphi(\delta)$, $0\le\delta\le\pi$ is a modulus of continuity of the second order if*

(a) $\varphi(0)=0$;

(b) $\varphi(\delta)$ *does not decrease and is continuous on* $[0,\pi]$;

(c) *in the expansion of φ by cosines* $\varphi(x)=A+\sum\limits_{n=1}^{\infty}a_n\cos nx$ $(0\le x\le\pi)$ *all the coefficients are positive,* $a_n\le 0$.

Proof. Note that expansion (c) certainly exists, because in view of (b) the function φ has a bounded variation on $[0,\pi]$, Furthermore, by virtue of (a), from (c) we have $A=-\sum\limits_{n=1}^{\infty}a_n$, therefore

$$\varphi(x)=-\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}a_n\cos nx=(-2)\sum_{n=1}^{\infty}a_n\sin^2 n\frac{x}{2},\quad 0\le x\le\pi. \tag{1}$$

In the capacity of the required function f_φ now we can consider

$$f_\varphi(x)=\frac12\sum_{n=1}^{\infty}a_n\cos nx,\quad -\infty<x<+\infty. \tag{2}$$

Since $f_\varphi(x)=(\varphi(x)-A)/2$, $0\le x\le\pi$, it follows that $f_\varphi\in C_{2\pi}$. For this function (see, e.g., [2], p. 70) we have

$$\Delta_h^2 f_\varphi(x)=(-2)\sum_{n=1}^{\infty}a_n\cos n(x+h)\sin^2 n\frac{h}{2}.$$

[1] For $f_\varphi\in C_{2\pi}$ we take the even extension $\varphi(\delta)/2$ of $(-\infty,+\infty)$.